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NOTE

GEURST'S STABILITY CRITERION FOR CONCENTRATION WAVES IN BUBBLY FLOWS

L. VAN WIJNGAARDEN
Department of Mechanical Engineering, University of Twente, 7500 AE Enschede, The Netherlands

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1. INTRODUCTION

During recent years work on the stability of concentration waves in bubbly flows has been published by Geurst (1985, 1986) and Geurst & Vreenegoor (1988) while the present author and his colleagues have worked on the same problem (van Wijngaarden & Biesheuvel 1988; Kok 1988; Biesheuvel & Gorissen 1990; van Wijngaarden & Kapteyn 1990).

Since results by these two groups are different it is of interest to inspect these differences.

We shall show in this note that the stability criterion, derived by Geurst (1985, 1986), is restricted to the case in which there are no interactions between bubbles. In the formulation by Geurst there is the added mass coefficient m, which is a function of the gas concentration ϵ. Geurst finds, under marginal stability, this to behave as

$$m(\epsilon) = \tfrac{1}{2}\rho_L \epsilon(1-\epsilon)(1-3\epsilon). \qquad [1]$$

While this is in itself correct, under the envisaged circumstances, the frame considered being a laboratory frame and bubbles having all the same velocity, some conclusions drawn from [1] by Geurst are erroneous. Namely, conclusions regarding bubble interactions, the probability distribution and special events at $\epsilon = 1/3$. It is the purpose of this note to show the exact meaning of [1].

2. PHYSICAL INTERPRETATION OF THE STABILITY OF VOID FRACTION WAVES

Imagine a void fraction profile like the one sketched in figure 1, where ϵ increases in the direction of x. The work by van Wijngaarden & Kapteyn (1990) admits the following physical explanation of stability. Let us write the drag $\mathbf{W}$ on a bubble as

$$\mathbf{W} = F(\epsilon)(\mathbf{U}_G - \mathbf{U}_0). \qquad [2]$$

Here $\mathbf{U}_G$ is the average gas velocity and $\mathbf{U}_0$ is the volume velocity. With the average liquid velocity $\mathbf{U}_L$, the volume velocity is defined as

$$\mathbf{U}_0 = \epsilon\mathbf{U}_G + (1-\epsilon)\mathbf{U}_L = \mathbf{U}_L + \epsilon(\mathbf{U}_G - \mathbf{U}_L). \qquad [3]$$

$F(\epsilon)$ is a drag coefficient. Experimental observation shows that $F(\epsilon)$ increases with increasing ϵ,

$$\frac{dF}{d\epsilon} > 0. \qquad [4]$$

If the available force to overcome friction is buoyancy, this means that $(\mathbf{U}_G - \mathbf{U}_0)$ will decrease with increasing ϵ.

Since small disturbances travel in the x direction, as a result of the volume conservation

$$\frac{\partial\epsilon}{\partial t} + U_G\frac{\partial\epsilon}{\partial x} + \epsilon\frac{\partial U_G}{\partial x} = 0, \qquad [5]$$

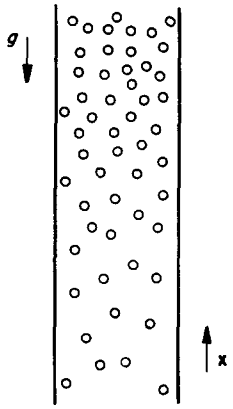

Figure 1. Bubbly flow in a vertical tube. The void fraction increases in the direction of x.

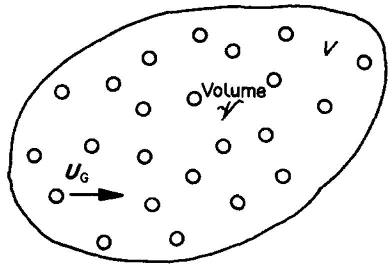

Figure 2. N bubbles in a volume V. The liquid occupies $V_L = V - \sum_{i=1}^{N} \mathscr{V}$.

with a velocity close to U_G, disturbances originating in a low ϵ region travel faster than those coming from a high ϵ region. This tends to cause instability. There is a stabilizing mechanism, however, due to the impulse of the fluid which is associated with added mass. The fluid impulse has contributions from the average bubble motion and from the fluctuations. We restrict this note to the former. The impulse of a bubble can on the average be represented as

$$\mathbf{I} = M(\epsilon)(\mathbf{U}_G - \mathbf{U}_0), \qquad [6]$$

where M is the added mass which is a function of ϵ. When, with increasing ϵ, the impulse of the liquid also increases, then the reaction force on the bubbles in figure 1 is negative in the x direction and is stabilizing. One could say that of the available force to overcome friction [in the case of figure 1, buoyancy] a part is taken off by this reaction force. Of course this is only the case when the impulse increases with increasing ϵ. In the opposite case, when the fluid impulse associated with the bubble motion decreases, the bubbles receive an extra upward force which is destabilizing.

Mathematical analysis, e.g. in van Wijngaarden & Biesheuvel (1988) and Biesheuvel & Gorissen (1990), shows that the condition for stability is

$$\frac{1}{M(\epsilon)} \frac{\mathrm{d}M}{\mathrm{d}\epsilon} > \frac{1}{F(\epsilon)} \frac{\mathrm{d}F(\epsilon)}{\mathrm{d}\epsilon}. \qquad [7]$$

When fluctuations and other effects like diffusion are taken into account, [7] becomes more complicated. For the present purposes this simplified picture suffices.

The ϵ dependence of both F and M is the result of hydrodynamic interactions between the bubbles. In the (non-realistic) case in which a change in ϵ would not produce a change in U_G, [5] indicates that small disturbances travel locally all with the same speed U_G. This is the situation which is envisaged by Geurst (1985, 1986). In his work concentration waves all travel with the same speed. This amounts in terms of [7] to $\mathrm{d}F/\mathrm{d}\epsilon = 0$. Hence, for that particular case the condition for marginal stability is

$$\frac{\mathrm{d}M}{\mathrm{d}\epsilon} = 0. \qquad [8]$$

This is, as we shall show in the following, precisely what result [1] by Geurst means.

3. STABILITY ANALYSIS OF GEURST (1986)

Geurst (1986) starts with a variational formulation, in which, of course, the kinetic energy plays a vital part. Kinetic energy takes different forms in different frames of reference. Geurst chooses a laboratory frame and writes the kinetic energy density, i.e. the kinetic energy in a unit volume of a bubbly suspension as in figure 2 as (ρ_L being the liquid density):

$$T = \tfrac{1}{2}\rho_L(1-\epsilon)U_L^2 + \tfrac{1}{2}\rho_L m(\epsilon)[(|\mathbf{U}_G - \mathbf{U}_L|)^2]. \qquad [9]$$

When $\epsilon = 0$ the only kinetic energy is $\frac{1}{2}\rho_L U_L^2$, for non-zero ϵ the presence of the bubbles leads to the additional energy represented by the second term on the r.h.s. of [9]. This, of course, is a valid representation. The difficulty is that Geurst identifies $m(\epsilon)$ in [9] as the average added mass density. This is not true, see section 4.

Using the Lagrangian formalism and ignoring friction, Geurst derives the Euler equations associated with the appropriate variational principle. He goes on to inspect the characteristics of these equations and finds that they are not always real. A minimum condition, and with that a condition for marginal stability is [1], which we repeat here for convenience:

$$m(\epsilon) = \tfrac{1}{2}\rho_L \epsilon(1-\epsilon)(1-3\epsilon). \qquad [10]$$

Geurst discusses this result more than once in his work.

In Geurst (1986, p. 463) he points out, referring to the calculation of $M(\epsilon)$ in van Wijngaarden (1976), that [10] requires a special configuration of bubbles, notably one in which gas bubbles tend to align themselves in the direction of mean flow. In the same paper, on the same page, he also suggests that the disappearance of $m(\epsilon)$ at $\epsilon = 1/3$ has to do with the transition to plug flow.

4. KINETIC ENERGY AND ADDED MASS DENSITY

The content of this section is a condensed version of Kok (1988). We look first at the motion of an isolated bubble moving through a liquid which far from the bubble has the velocity $\mathbf{U}_0$. The motion is incompressible and irrotational. We consider the kinetic energy of the liquid in a frame which moves with the velocity $\mathbf{U}_0$:

$$T = \tfrac{1}{2}\rho_L \int (\mathbf{u} - \mathbf{U}_0)\cdot(\mathbf{u} - \mathbf{U}_0)\mathrm{d}V. \qquad [11]$$

Writing $\mathbf{u} - \mathbf{U}_0 = \nabla\phi$ this can be written with help of Gauss's theorem as

$$\tfrac{1}{2}\rho_L \int_\infty \varphi\nabla\varphi \cdot \mathbf{n}\mathrm{d}A - \tfrac{1}{2}\rho_L \int_{\text{bubble}} \varphi\nabla\varphi \cdot \mathbf{n}\mathrm{d}A. \qquad [12]$$

Since at infinity the velocity $\mathbf{u} - \mathbf{U}_0 = \nabla\phi$ vanishes fast enough, the first integral in [12] is zero and we have

$$T = -\tfrac{1}{2}\rho_L \int_{\text{bubble}} \phi\nabla\phi \cdot \mathbf{n}\mathrm{d}A = -\tfrac{1}{2}(\mathbf{U}_G - \mathbf{U}_0)\cdot \int \rho_L \phi \mathbf{n}\,\mathrm{d}A. \qquad [13]$$

The integral in [13] is precisely the impulse

$$-\int \rho_L \phi \mathbf{n}\,\mathrm{d}A = \mathbf{I} = M(\mathbf{U}_G - \mathbf{U}_0). \qquad [14]$$

Hence, in this case the quantity $\rho_L K$ in

$$T = \tfrac{1}{2}\rho_L(|\mathbf{U}_G - \mathbf{U}_0|)^2 K \qquad [15]$$

is the same as the quantity M in

$$\mathbf{I} = M(\mathbf{U}_G - \mathbf{U}_0). \qquad [16]$$

One question is whether this is also true for the average of similar quantities in a mixture. Another question is whether this holds when we describe the energy in a laboratory frame.

Let us first address the former question. When there are, for instance in the cloud in figure 2, N bubbles in a cloud of volume V, we define in analogy with [11] the kinetic energy density

$$T = \frac{\frac{1}{2}\rho_L}{V} \int_{V_L} (\mathbf{u} - \mathbf{U}_0)\cdot(\mathbf{u} - \mathbf{U}_0)\mathrm{d}V, \qquad [17]$$

where V_L is the volume occupied by liquid, or

$$T=\sum_{i=1}^{N}-\frac{\frac{1}{2}\rho_L}{V}\int_{A_i}\phi\nabla\phi\cdot\mathbf{n}\,\mathrm{d}A=\sum_{i=i}^{N}-\frac{1}{2}\left[\frac{\mathbf{u}_{G,i}-\mathbf{U}_0}{V}\right]\cdot\int\rho_L\phi\mathbf{n}\,\mathrm{d}A,$$

where $\mathbf{u}_{G,i}$ indicates the velocity of the ith bubble.

Comparison with [13] and [14] shows that the coefficients in the kinetic energy and in the impulse are the same when (angular brackets meaning averages):

$$\left\langle(\mathbf{u}_G-\mathbf{U}_0)\int\rho_L\phi\mathbf{n}\,\mathrm{d}A\right\rangle=(\mathbf{U}_G-\mathbf{U}_0)\left\langle\int\rho_L\phi\mathbf{n}\,\mathrm{d}A\right\rangle. \qquad [18]$$

This is only the case if there are no fluctuations in the gas bubble velocities. Since fluctuations are brought about by interactions, this is the case when no interactions take place. The only case in which there are interactions and no fluctuations occurs when there is a long range order in the bubbly suspension, leading to a periodic array. In such a case, the proper expansion parameter to describe interaction effects is $\epsilon^{1/3}$, not ϵ.

For further reference, and in analogy with [15], we write [17] as

$$T=\frac{1}{2}\rho_L(1-\epsilon)\frac{1}{V_L}\int_V(\mathbf{u}-\mathbf{U}_0)\cdot(\mathbf{u}-\mathbf{U}_0)\mathrm{d}V=-\sum\frac{1}{2}\frac{(\mathbf{u}_{G,i}-\mathbf{U}_0)}{V}\int\rho_L\phi\mathbf{n}\,\mathrm{d}A$$

$$=\frac{1}{2}\rho_L K^*(\epsilon)(|\mathbf{U}_G-\mathbf{U}_0|)^2. \qquad [19]$$

When we write for the impulse density

$$\mathbf{I}=\rho_L M^*(\epsilon)(\mathbf{U}_G-\mathbf{U}_0), \qquad [20]$$

we conclude that $M^*(\epsilon)$ and $\rho_L K^*(\epsilon)$ are not the same in general but are identical when all bubbles have the same velocity. This is the case in the situation envisaged by Geurst (1985, 1986).

To arrive at energy or impulse density there is a summation over N bubbles and a division by V, the volume of the bubbly cloud, i.e. a multiplication by N/V, the number density n, say. Since

$$\epsilon=n\mathscr{V}, \qquad [21]$$

$\mathscr{V}$ being the volume of an individual bubble, we have $n=\epsilon/\mathscr{V}$. Therefore, we find from $M^*(\epsilon)$ in [20] or $\frac{1}{2}\rho_L K^*(\epsilon)$ in [19] the average quantity pertaining to an individual bubble through multiplication by $\mathscr{V}/\epsilon$.

The average kinetic energy associated with one bubble is then

$$\langle T\rangle=\frac{1}{2}\rho_L K^*(\epsilon)\frac{\mathscr{V}}{\epsilon}(|\mathbf{U}_G-\mathbf{U}_0|)^2 \qquad [22]$$

and

$$\langle\mathbf{I}\rangle=M^*(\epsilon)\frac{\mathscr{V}}{\epsilon}(\mathbf{U}_G-\mathbf{U}_0). \qquad [23]$$

Because in the case envisaged by Geurst all bubbles move with the same velocity there is equivalence between $\frac{1}{2}\rho_L K^*(\epsilon)$ and $M^*(\epsilon)$ and he should have found that his $m(\epsilon)\mathscr{V}/\epsilon$ is independent of ϵ. We see from [1] that this is not the case. What is the reason for this?

5. ENERGY DENSITY IN A LABORATORY FRAME

We express the kinetic energy T of the mixture in the laboratory frame as

$$T'=-\tfrac{1}{2}\rho_L\int\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}V. \qquad [24]$$

The integration is over the whole volume V occupied by the cloud. The corresponding energy density is obtained by dividing through V.

Since only the liquid contributes, the gas density being negligibly small,

$$T = \frac{1}{2}\rho_L \frac{1}{V}\int_{V_L} \mathbf{u}\cdot\mathbf{u}\,dV = \frac{1}{2}\rho_L(1-\epsilon)\frac{1}{V_L}\int_{V_L}\mathbf{u}\cdot\mathbf{u}\,dV, \qquad [25]$$

where, again, V_L is the volume occupied by the liquid. Again $\mathbf{u}$ can be expressed in terms of a velocity potential, however we cannot reduce the integration from volume to surface integration over bubbles only since the integral over the surface at large distance diverges. Therefore, we write

$$\mathbf{u} = \mathbf{u} - \mathbf{U}_0 + \mathbf{U}_0 = \mathbf{u}' + \mathbf{U}_0 \qquad [26]$$

and insert this into the integral in [25].

Making use of [3] and of

$$\frac{1}{V_L}\int_{V_L}\mathbf{u}'\cdot\mathbf{U}_0\,dV = \mathbf{U}_0\cdot(\mathbf{U}_L - \mathbf{U}_0) = -\epsilon(\mathbf{U}_G - \mathbf{U}_L)\cdot\mathbf{U}_0,$$

we obtain

$$T = \frac{1}{2}\rho_L(1-\epsilon)U_L^2 - \frac{1}{2}\rho_L\epsilon^2(1-\epsilon)(|\mathbf{U}_G - \mathbf{U}_L|)^2 + \frac{1}{2}\rho_L\frac{1}{V}\int_{V_L}\mathbf{u}'\cdot\mathbf{u}'\,dV.$$

The third term on the r.h.s. has been calculated in section 4, resulting in [19]. Making use of

$$\mathbf{U}_G - \mathbf{U}_0 = (1-\epsilon)(\mathbf{U}_G - \mathbf{U}_L),$$

we finally obtain, using [19]:

$$T = \tfrac{1}{2}\rho_L(1-\epsilon)U_L^2 + \tfrac{1}{2}(|\mathbf{U}_G - \mathbf{U}_L|)^2[\rho_L K^*(\epsilon)(1-\epsilon)^2 - \rho_L\epsilon^2(1-\epsilon)]. \qquad [27]$$

When interactions are disregarded $\rho_L K^*(\epsilon) = M^*(\epsilon)$, see [23], which is connected with the average added mass $M^*\mathcal{V}/\epsilon$. Hence, the expression

$$T = \tfrac{1}{2}\rho_L(1-\epsilon)U_L^2 + \tfrac{1}{2}(|\mathbf{U}_G - \mathbf{U}_L|)^2[M^*(\epsilon)(1-\epsilon)^2 - \rho_L\epsilon^2(1-\epsilon)] \qquad [28]$$

can be compared with Geurst's expression [9].

We see now that, in contradiction to Geurst's conclusion, it does not have the added mass density as a coefficient in the term quadratic in $(|\mathbf{U}_G - \mathbf{U}_L|)^2$ but

$$M^*(\epsilon)(1-\epsilon)^2 - \epsilon^2\rho_L(1-\epsilon). \qquad [29]$$

In the absence of interactions, there are no fluctuations in the velocities of individual bubbles which therefore all move with the same velocity. The added mass of each bubble is then $\frac{1}{2}\rho_L\mathcal{V}$, and hence

$$M^*(\epsilon) = \frac{\epsilon}{\mathcal{V}}\cdot\tfrac{1}{2}\rho_L\mathcal{V} = \tfrac{1}{2}\rho_L\epsilon.$$

Inserting this into [29] gives

$$\tfrac{1}{2}\rho_L\epsilon(1-\epsilon)(1-3\epsilon),$$

exactly the quantity (see [9]) which Geurst finds.

6. CONCLUSION

We must conclude that Geurst's result [9] means that with bubbles at a velocity unaffected by changes in ϵ concentration waves are marginally stable when the average added mass does not depend on concentration. Geurst, on the contrary, states that [9] describes a specific dependence of average mass on ϵ, necessary for marginal stability, and connects this with statements about bubble configurations and singular behaviour at $\epsilon \sim 1/3$.

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